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Self-organization in populations of competing agents

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A population of heterogenous agents competing through a minority rule is investigated. Agents which frequently lose are selected for evolution by changing their strategies. The stationary composition of the population resulting from this self-organization process is computed analytically. Results are compared with numerical simulations of two different minority games and other analytical treatments available in the literature.

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The mechanisms under which a population of competing agents self-organizes is a problem that has gained a lot of interest among the physics community in the last few years [1–7]. As in traditional statistical mechanics, the main goal is to determine the average behavior of the population based on elementary rules which characterize the competition among agents. Given the actual state of art most of the works have been devoted to the characterization, by means of numerical experiments or analytical treatments, of the different models proposed in the literature [1–3,7], while a general description is still missing.

This work is an attempt to extract some general features present in the subclass of models where the performance of the agents is characterized by a certain minority rule [1,2,7]. Agents are assumed to be heterogeneous and the main goal is to obtain the final composition of the population if a certain evolution mechanism is introduced. The agents will be supposed not to be subscribed to any regular lattice which rules out the existence of spatial correlations allowing a mean field (MF) treatment.

The work is organized as follows. First the general features of the class of models under consideration are introduced. Based on these rules the MF rate equations describing the evolution of the population are derived and the stationary solutions are computed. These results are compared with numerical simulations of two particular minority "games." Moreover, a comparison with the analytical results already available for the model by Johnson *et al.* [3–6] is also drawn. In all cases a very good agreement is obtained.

Consider a population of N (odd) heterogeneous Boolean agents characterized by a certain property x. The Boolean nature of the agents restricts their action to only two possibilities. An example is the Arthur-bar problem [1] in which each agent attends or not a bar based on the past attendance to the bar. Another example is the minority game introduced by Challet and Zhang [2], in which the agents can either buy or sell. In both models a good choice of x is the probability that an agent takes different decisions in two different steps. A different choice is taken in the model introduced by Johnson *et al.* [3]. In this case x is the probability that an agent accepts the decision suggested by its strategy or does the opposite. In general one may think of x as any property, or set of properties, which distinguish the different strategies one agent can choose.

On each step each agent takes one of the two decisions, and those being in the minority win. The long time performance of the agents will be characterized by a cumulative index z in such a way that each time an agent wins it is rewarded with a positive point ($z \to z+1$); otherwise, it is punished, and receives a negative point ($z \to z-1$). Evolution in the population is introduced assuming that each time the performance z of an agent goes below a threshold $-z_c$ ($z_c > 0$) it is selected for evolution and changes its strategy: a new value of x ($x \to x'$) is selected among a certain distribution $P_0(x')$ and its performance is reset to zero ($z \to 0$).

A similar evolutionary mechanism has been already considered in [3]. One also may think in other implementations such as the extremal rules considered in [2,7], or a probabilistic rule in which agents can change strategy even below z_c according to certain probability, which in general may depend on z. In any case it is expected that all of them give the same qualitative behavior in the large z_c limit.

Having defined the model let us determine which is the stationary composition of the population of agents $P(x)$, the fraction of agents of type x. For doing so we first derive the rate equations which describe the dynamical evolution of $P(x)$, which in general can be written as

$$\frac{\partial}{\partial t} P(x) = \sum_{x'} [W(x',x)P(x') - W(x,x')P(x)], \qquad (1)$$

where $W(x',x)$ is the transition rate from x' to x.

The only rule which allows a change in x is the evolution rule, in which the agent chooses a new value of x', selected with probability $P_0(x')$. Thus, if $\lambda(x)$ is the fraction of x agents which are changing their strategy per unit time, then the transition rates are given by $W(x,x') = \lambda(x)P_0(x')$. After substitution of this expression into Eq. (1) it results that

$$\frac{\partial}{\partial t} P(x) = P_0(x) \sum_{x'} \lambda(x')P(x') - \lambda(x)P(x). \qquad (2)$$

To determine $\lambda(x)$ we need to consider the temporal evolution of the cumulative performance index $z(x)$. Let $P_m(x)$ be the probability per unit time that an x agent is in the minority in one step. Thus, with probability $P_m(x)$ [$1 - P_m(x)$] the performance increases (decreases) by one unit. More generally one can define $P_m(x)$ as the probability that an agent wins a point, regardless how it did. On the other hand, if an agent changes strategy, which happens with probability $\lambda(x)$, then its performance increases by z_c. These elementary processes lead to the rate equation

$$\frac{\partial}{\partial t} z(x) = 2P_m(x) - 1 + z_c\lambda(x). \tag{3}$$

Provided $P_m(x) < 1/2$ for all x the system described by Eqs. (2) and (3) will always reach a stationary state. In this state time derivatives vanish, obtaining

$$\lambda(x) = \frac{1}{z_c}[1 - 2P_m(x)], \tag{4}$$

$$P(x) = \frac{A}{1 - 2P_m(x)} P_0(x), \tag{5}$$

where A is a normalization constant.

Agents with $P_m(x)$ close to 1/2 are more stable to change in strategy, as can be seen from Eq. (4) and, therefore, they will have a larger participation ratio in the stationary population, as follows from Eq. (5). The distribution from which the value of x is extracted is thus modulated by the probability of being in the minority, yielding the stationary population in Eq. (5).

On the other hand, from Eq. (4) it can be seen that in the limit $z_c \to \infty$ the fraction of agents changing strategy becomes infinitesimal and, therefore, in this limit one can expect to obtain the same results as if one used an extremal evolution rule. Another aspect to be emphasized is the fact that the parameter z_c, which in principle is the only evidence of the particular evolution rule chosen here, does not appear in the expression for the stationary distribution of agents in Eq. (5) and, therefore, this result is expected to hold independent of the particular evolution rule chosen.

So far the results obtained here are very general and expected to apply to a wide class of models of Boolean agents with the minority rule. To go further we have to determine $P_m(x)$, which may, however, depend on the particular model under consideration. Below two different cases are analyzed. One is a very simple model of noninteracting agents (NIA), which make their decision at random. The other is the very recent implementation of a population of Boolean agents built up onto a Kauffman's network (AKN) introduced by Paczuski *et al.* [7].

In the AKN model introduced in [7] agents makes their decisions based on the previous decision of some other agents. A Boolean variable σ_i ($i = 1,2,\ldots,N$; $\sigma_i = 0,1$) is assigned to each agent which is representative of the two possible decisions one agent may take. The decision taken by each agent is based on the decision taken by other K agents chosen at random ($i_1,\ldots,i_K$) in the previous step, according to certain Boolean function f_i selected at random among the set of all the 2^{2^K} possible Boolean functions with K inputs, i.e.,

$$\sigma_i(t+1) = f_i[\sigma_{i_1}(t),\ldots,\sigma_{i_K}(t)]. \tag{6}$$

Without lost of generality and for a reason which becomes clear later, Boolean functions which give the same output independent of the inputs are ruled out.

In the original variant of this model [7] an extremal evolution rule is used, such that after a certain time the worst agent is selected for evolution. This rule is relaxed here with the barrier evolution rule considered previously, which yields the extremal dynamics as $z_c \to \infty$. Moreover, one can see that a property that makes differences among the agents can be the number of times 1 appears on the $M = 2^K$ outputs of its Boolean function, denoted by n. If n is close to $M/2$ the agent will give as output 0 or 1, depending on the configuration of its K neighbors, with approximately the same probability. Otherwise, if n is close to 1 or M the agent will practically give the same output independent of the configuration of its neighbors.

The NIA model is a simplification of the above model in which interactions among agents are ruled out. In this case on each step an n agent gives 1 as output with a probability n/M and 0 otherwise. For this case the agent's decision does not depend on decisions of any other agents and, therefore, there is no other correlation than the one introduced by the minority rule, which depends on the output of all agents.

Numerical simulations were performed for a population of $N = 99$ agents and $K = 2,3,4,5$. In all cases the system is updated until it reaches the stationary state and then the average is taken over the temporal evolution of the population. The resulting data is averaged over different realizations of the initial Boolean functions and over the choice of neighbors in the interacting case.

Since the new Boolean functions are selected at random one has

$$P_0(n) = B(n;0.5,M) \Big/ \sum_{r=1}^{M-1} B(r;0.5,M), \tag{7}$$

where $B(n;p,m)$ is the binomial distribution, the probability to obtain 1 n times and 0 $m-n$ times given on each single event 1 happens with probability p. The values $n = 0$ and M are ruled out because they give fixed strategies, and consequently the binomial distribution is renormalized.

To compute the stationary composition $P(n)$ one has to compute the probability $P_m(n)$ that an agent of type n is in the minority, and then plug in the result in Eq. (5). For N large the game is expected to be symmetric in the sense that with probability 1/2 the minority is the group of agents which takes the 1 (or 0) as output. In such a case a fixed agent will, in a long time window, have a probability 1/2 of being in the minority while agents changing their output very often are expected to have a lower probability to be in the minority.

The main hypothesis taken here is that $1 - 2P_m(n) = f[\rho(n)]$, where $f[\rho(n)]$ is a smooth function of the probability per unit time $\rho(n)$ that an agent changes its output. Since fixed players ($\rho = 0$) has a probability 1/2 to be in the minority, then $f(0) = 0$. In the following $f(\rho)$ is expanded around $\rho = 0$, keeping only the linear term, resulting in

$$1 - 2P_m(n) \propto \rho(n) + O[\rho(n)^2]. \tag{8}$$

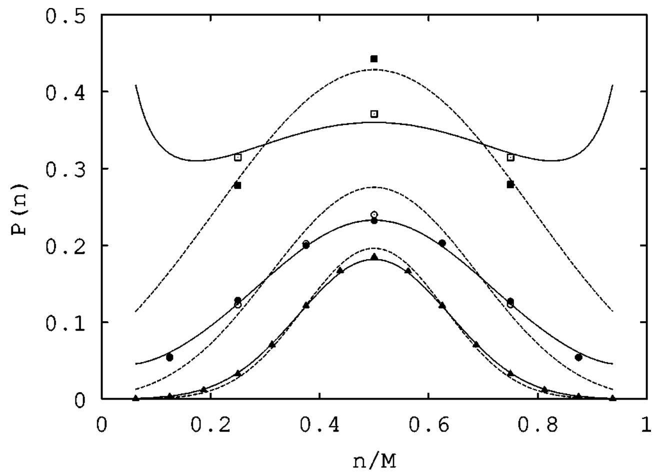

FIG. 1. Stationary composition of the population of agents. The points were obtained from numerical simulations of a population of $N=99$ agents with a threshold $z_c=100$ and $K=2$ (squares), $K=3$ (circles), and $K=4$ (triangles). In all cases the open and closed symbols correspond to the NIA and AKN models, respectively. The solid lines are obtained using Eq. (10) with $B=0$ (dashed lines) and $B>0$ (solid lines) for the corresponding values of K.

Now, for NIA $\rho(n)$ is just the probability to find two different outputs on the agent's strategy, which is given by $P_c(n)=2(n/M)(1-n/M)$. For AKN one should also take into account that agents can change their output only if at least one of its inputs has changed output in the previous step. Within an MF approximation the last event happens with probability $K\overline{\rho}$. Thus, in general,

$$\rho(n)=BP_c(n), \tag{9}$$

where $B=1$ and $B=K\overline{\rho}$ for NIA and AKN, respectively.

By construction in both models $P_c(n)>0$ because the cases $n=0$ and M have been ruled out. Hence, $1-2P_m(n) \propto BP_c(n)$ can only be zero if $B=0$, and if does it is zero for all n. If the second possibility happens then all agents will have the same probability to be in the minority and, therefore, their distribution in the stationary state will be the distribution from where the Boolean functions are extracted, i.e., $P(n)=P_0(n)$. Hence, Eqs. (5), (8), and (9) yield the following alternative:

$$P(n)=\begin{cases} \dfrac{\overline{P_c}}{P_c(n)}P_0(n) & \text{for } B>0 \\ P_0(n) & \text{for } B=0, \end{cases} \tag{10}$$

where $\overline{P_c}=\Sigma_n P_c(n)P(n)$, which is the final output of the present calculation.

For NIA as mentioned above $B=1$ and, therefore, the alternative $B>0$ takes place. The comparison of this prediction with numerical data is shown in Fig. 1. The agreement is quite well for all values of K, proving that the ansatz in Eq. (9) applies for this model.

For AKN $B=K\overline{\rho}$ and one should analyze whether $\overline{\rho}$ is zero or not. As is well known the Kauffman's network displays qualitatively different behavior depending on the value of K [8]. For $K\leqslant 2$ and independent of the initial conditions the network evolves to a frozen configuration with $\overline{\rho}=0$. On the contrary for $K>2$ the network evolves to periodic orbits with periods growing exponentially with N (''chaotic phase''), in which $\overline{\rho}>0$.

Thus, the population of agents building up onto the Kauffman's network is a very good scenario to test the validity of Eq. (10) because both alternatives can be observed. The comparison is shown in Fig. 1. For $K=2$ it can be seen that the numerical data is in better agreement by $P(n)=P_0(n)$, as predicted above. On the contrary, for $K>2$ the data is in better agreement with the case $B>0$ in Eq. (10).

For $K>2$ both the NIA and AKN models yield the same distribution $P(n)$ and, therefore, the correlations introduced by the network are in those cases irrelevant. Moreover, with increasing K the composition of the population gradually approaches $P_0(n)$, which explains the lost of self-organization observed in [7].

Finally the model of Johnson *et al.* [3] is considered. In this case the decisions taken by the agents depend on the previous history of the winning group. For each agent it is available in the information of which has been the winning group $\sigma_w(t)$ [$\sigma_w(t)=0,1$] in the last K steps. Moreover, to each of them a strategy f_i is assigned, which is extracted at random among all the possible Boolean functions of K inputs. Then, for each possible past history each agent will give a well defined output based on its own strategy, i.e.,

$$\sigma_i(t+1)=f_i[\sigma_W(t), \ldots, \sigma_W(t-K+1)]. \tag{11}$$

Notice that in this case the output of each agent depends on global information and not on the outputs of any other agent, as in the AKN model [see Eq. (6)].

So far this model is just a variant of the minority game of Challet and Zhang [2] for the case in which each agent has only one strategy at his disposal. In the variation by Johnson *et al.* [3] a probability p_i is assigned to each agent based on whether or not he accepts the output of its strategy. With probability p_i $(1-p_i)$ he uses the outcome (the opposite outcome) of its strategy. Moreover, they introduced the cumulative index z_i to measure the performance of each agent in a similar way as described above. The only difference is that for this model when z_i goes below the threshold, $-z_c$ the agent does not change its Boolean function f_i but rather its probability p_i, choosing a new one at random in the interval $[p_i-R/2,p_i+R/2]$ with reflective boundary conditions.

Some analytical results are already available for this model [4,6]. Using a diffusionlike approach [4] or a detailed probabilistic calculation [6] it has been shown that in the stationary population it has the following composition:

$$P(p)=\frac{A}{1-2P_m(p)}, \tag{12}$$

where A is a normalization constant and $P_m(p)$ is the probability that an agent of type p is in the minority [9].

Equation (12) is actually quite similar to Eq. (5), with the choice $x=p$. Since for this model the new values of p are extracted from a uniform distribution it is expected that $P_0(p)$ does not depend on p. Hence, Eq. (12) can be seen as a limiting case of Eq. (5) when applied to the particular evolution rule of the model by Johnson *et al.* [3], where $P_0(x) = \text{const}$.

In order to go beyond this result one has to determine $P_m(p)$. This has already been done in [6] resulting that for large N, $1-2P_m(p) \approx C(N)2p(1-p)$, where $C(N) \sim N^{-1/2}$. This results does not seem to have any relation with the ansatz in Eq. (10). However, one should notice that $2p(1-p)$ is just the probability $P_c(p)$ that an agent of type p gives two different outputs, given the output of its Boolean function has remained fixed. Therefore, from this result and Eq. (12) it follows that

$$P(p) = \frac{\overline{P_c}}{P_c(p)}. \tag{13}$$

This equation is just the $B > 0$ alternative of Eq. (10). Hence, the functional dependence of $P_m(x)$ and $P(x)$ on $P_c(x)$ appears to be universal for the class of models studied here.

In summary, a heterogeneous population of competing agents has been studied. Based on general arguments, such as the minority rule and evolution, the stationary composition of the population of agents has been computed as a function of the probability $P_m(x)$ of being in the minority. Further analysis reveals that the relation $1-2P_m(x) \propto P_c(x)$ is universal for this class of models, where $P_c(x)$ is the probability that an agent of type x gives different outputs.

For the AKN it is concluded that except for $K=2$ the correlations introduced by the network are irrelevant and agents can be considered to be independent. In the particular case $K=2$, which corresponds to the critical network, the population buildup onto the Kauffman's network reaches a stationary state in which all agents have a probability 1/2 to be or not in the minority. Moreover, the loss of self-organization with increasing K was shown to take place gradually.

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[1] W. B. Arthur, Am. Econ. Rev. **84**, 406 (1994).
[2] D. Challet and Y.-C. Zhang, Physica A **246**, 407 (1997); **256**, 514 (1998).
[3] N. F. Johnson, P. M. Hui, R. Jonson, and T. S. Lo, Phys. Rev. Lett. **82**, 3360 (1999).
[4] R. D'hulst and G. J. Rodgers, Physica A **270**, 514 (1999).
[5] H. Ceva, Physica A **277**, 496 (2000).
[6] T. S. Lo, P. M. Hui, and N. F. Johnson, Phys. Rev. E **62**, 4393 (2000).
[7] M. Paczuski, K. E. Bassler, and Á. Corral, Phys. Rev. Lett. **84**, 3185 (2000).
[8] S. A. Kauffman, *The Origin of Order* (Oxford University Press, New York, 1993).
[9] A different notation is used in [4,6] in which $P_m(p)$ is written as $\tau(p)$.